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Translated by L.K.

PMM U.S.S.R.,Vol.54,No.2,pp.285-289,1990
Printed in Great Britain

0021-8929/90 $10.00+0.00
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THE CONSTRUCTION OF THE CONSTANT-VELOCITY CONTOUR OF A FOUNDATION OF A HYDRAULIC INSTALLATION IN THE CASE OF THE FILTRATION OF TWO LIQUIDS OF DIFFERENT DENSITY*

E.N. BERESLAVSKII

An underground, constant-velocity contour is constructed for the case when a layer of stagnant salt water forms at a certain depth in a flow of water under a dyke. Results of numerical computations are presented and an analysis given of the influence of the fundamental defining parameters of a model on the form and size of the underground contour of a dam. Limiting cases of flows are mentioned, namely the scheme with a water-confining stratum /1/ and filtration around a point channel /2-4/.

1. Formulation of the problem. Consider the steady plane flow of fresh water of density ρ_1 under an underground impermeable contour of a channel BC in the case when a layer of salt water of density $\rho_2\,(\rho_2>\rho_1)$ appears at a certain depth above an impermeable layer of salt. The domain of filtration z (Fig.1) is bounded from below by a boundary AD passing through a fixed point $z_0=-ih_0$ where h_0 is the depth of the initial surface (before the squeeze) of salt water. The pressure H acting on the installation and the width of the flood bed l, whose left-hand end is fixed at the point $B\,(z=-l_1)$ are assumed given, and the boundaries of the head and tail by AB and CD are horizontal. The flow obeys D'Arcy's law, and the soil is assumed to be homogeneous and isotropic.

Let us introduce the complex potential $\omega=\varphi+i\psi$ and complex coordinate $z=x+iy$ referred, respectively, to $\varkappa h_0$ and h_0, where $\varkappa$ is the soil filtration coefficient. Let us put $\varphi=-H/2$ on $AB, \varphi=H/2$ on CD and $\omega=Q$ along the water-impermeable contour of the flood bed BC, where Q is the filtration flow rate. Then we find that the following conditions must hold at the boundary line AD:

$$\varphi-cy=\text{const},\ \psi=0\ (c=\rho_2/\rho_1-1) \tag{1.1}$$

The first relation of (1.1) for the segment AD follows from the assumption that salt water is stagnant and the pressure remains continuous during the passage across the boundary line /5, 6/. The condition of continuity of the potential at infinity to the left and right, together with condition /5, 6/ $h_0=(h_1+h_2)/2$, which follows from the assumptions concerning the incompressibility of the liquid, determine the value of the constant in condition (1.1), and the difference in depth to the left and right after squeezing $h_1-h_2=H/c$. From this it follows that

$$h_1=h_0+H/(2c),\ h_2=h_0-H/(2c) \tag{1.2}$$

and this determines the region of flow of the ground water.

Next it is required to construct an underground contour BC, so that the filtration rate

Prikl.Matem.Mekhan.,54,2,342-346,1990

along it has a constant value v_0, and we shall also need to determine the position of the boundary of separation AD.

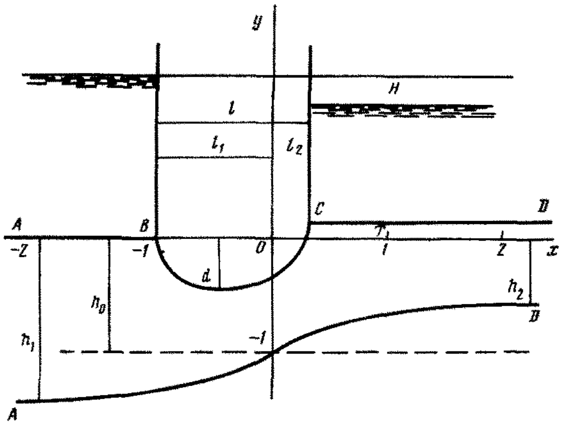

Fig.1

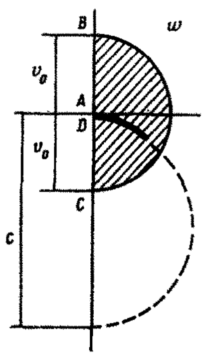

Fig.2

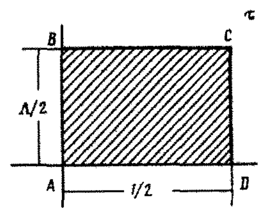

Fig.3

2. ***Constructing the solution.*** *The case* $v_0 < c$. The region of the complex velocity plane w shown in Fig.2 represents a circular pentagon with right angles at the corners A, B, C, D and the cut AD. We take, as the canonical region, the rectangle in the τ plane (Fig.3) connected to the ω-plane by the relation

$$\tau = \omega/(2H) + {}^1/_4 \tag{2.1}$$

Then the filtration flow rate will be given by the formula

$$Q = HK'/K = H\Lambda \tag{2.2}$$

where $K = K(k)$ is a complex elliptic integral of the first kind with the modulus k, $K' = K(k')$, $k' = \sqrt{1-k^2}$, $\Lambda = K'/K$.

Let us now map conformally the rectangle of τ-plane onto the region $ABCD$ of the w-plane using the method given in /7/ for constructing the mapping functions for circular polygons of the type discussed here. Taking (2.1) into account, we obtain the solution of the problem in the following parameteric form:

$$\frac{dz}{d\tau} = \frac{2H}{v_0}\,\frac{e^{-i\pi\beta}\vartheta_4(\tau+\alpha)+e^{i\pi\beta}\vartheta_4(\tau-\alpha)}{\vartheta_4(\tau+\alpha)-\vartheta_4(\tau-\alpha)} \tag{2.3}$$

$$\alpha = \operatorname{arctg}\sqrt{(c+v_0)/(c-v_0)}/\pi, \quad \beta = -\operatorname{arctg}\,(v_0/\sqrt{c^2-v_0^2})/\pi$$

where ϑ_4 is the theta function /8/.

Writing expression (2.3) for various segments of the boundary of the region τ and integrating, we obtain the equations for the corresponding segments of the scheme. Below we give the required equations of the boundary of separation AD and of the constant velocity contour BC:

$$x=\int_{1/4}^{\tau} X_{AD}(\tau)\,d\tau,\quad y=2H\tau/c-h_1 \quad \text{on } AD \tag{2.4}$$

$$0\leqslant\tau\leqslant 1/2,\quad X_{AD}=\operatorname{Re}(dz/d\tau)$$

$$x=-l_1+\int_0^{\tau} X_{BC}(\tau)\,d\tau,\quad y=\int_0^{\tau} Y_{BC}(\tau)\,d\tau \quad \text{on } BC \tag{2.5}$$

$$0\leqslant\tau\leqslant 1/2,\quad X_{BC}=\operatorname{Re}(dz/d\tau)|_{BC},\quad Y_{BC}=\operatorname{Im}(dz/d\tau)|_{BC}$$

$$\left(\frac{dz}{d\tau}\right)_{BC}=\frac{2H}{v_0}\,\frac{e^{-i\pi(\alpha+\beta)}\vartheta_1(\tau+\alpha)+e^{i\pi(\alpha+\beta)}\vartheta_1(\tau-\alpha)}{e^{-i\pi\alpha}\vartheta_1(\tau+\alpha)-e^{i\pi\alpha}\vartheta_1(\tau-\alpha)}$$

Let us note the limiting case of $c=\infty\ (\rho_2=\infty)$, which can be treated within the framework of the filtration scheme under discussion, as the "freezing" of salt water. The line of separation now becomes a horizontal, water confining stratum, and this can be confirmed using Eq.(2.3) with help of the expression for β, and remembering that $\psi=0$ on AD. As a result, we find that when $c=\infty$ on AD, then

$$\frac{dz}{d\tau}=\frac{2H}{v_0}\,\frac{\vartheta_4\left(\tau+\frac{1}{4}\right)+\vartheta_4\left(\tau-\frac{1}{4}\right)}{\vartheta_4\left(\tau+\frac{1}{4}\right)-\vartheta_4\left(\tau-\frac{1}{4}\right)} \tag{2.6}$$

and therefore $(\partial y/\partial\varphi)_{AD}=0,\ y_{AD}=\text{const.}$

Passing in (2.6) to the analytic functions /8/, we transform (2.6), after certain manipulations and using the substitution

$$\xi=\frac{2\lambda\,\mathrm{sn}^2(2K\tau,k)-(1+\lambda)}{\lambda[1+\lambda-2\,\mathrm{sn}^2(2K\tau,k)]},\quad \lambda=\frac{1-k'}{1+k'}$$

to a form which is completely identical with Eq.(7.8) of /6, p.189/, provided that we put in the latter $\alpha=0,\beta=1$, or with the result of dividing the equations (1.5) and (1.4) of /1/.

Thus we find that in the limiting case in question the boundaries of the bays are at the same height $(T=0)$ and the flood bed itself becomes symmetrical $(l_1=l_2)$.

Case $v_0=c$. Passing in (2.3) to the limit as $v_0\to c$, taking into account the formulas for α and β and removing the resulting indeterminacy 0/0 by means of L'Hopital's rule we obtain, after some reduction

$$\frac{dz}{d\tau}=\frac{2H}{cK}\,\frac{-\pi+iKZ\,[K(2\tau-1)]}{Z\,[K(2\tau-1)]} \tag{2.7}$$

where Z is the Jacobi zeta function /8/. We can obtain the same representation by inverting the region w relative to the circle with centre at the point $w=-iv_0$ which, in the present case, represents the point of intersection of all boundary segments of the region of complex velocity.

Case $v_0>c$. Transforming the solution (2.3) in conformity with the relation $v_0>c$, we obtain

$$\frac{dz}{d\tau}=\frac{2Hi}{v_0}\,\frac{e^{-\pi\beta}\vartheta_3(\tau+i\alpha)-e^{\pi\beta}\vartheta_3(\tau-i\alpha)}{\vartheta_3(\tau+i\alpha)-\vartheta_3(\tau-i\alpha)} \tag{2.8}$$

$$\alpha=\operatorname{arth}\sqrt{(v_0-c)/(v_0+c)}/\pi,\quad \beta=\operatorname{arth}\left(\sqrt{v_0^2-c^2}/v_0\right)/\pi$$

Let us mention the limiting case of $v_0=\infty$, connected with the degeneration of the region of complex velocity. The points B and C will then coalesce at infinity (Fig.2) and the rectangle in the τ-plane will be transformed into a half-strip (Fig.3). Here $k=0$ and from (2.8) we obtain

$$\frac{dz}{d\tau}=\frac{2H}{c}\,\frac{A+\cos 2\pi\tau+i\sin 2\pi\tau}{\sin 2\pi\tau} \tag{2.9}$$

where A is a constant regulating the position of the tip of the cut in the w-plane. Eq.(2.9) is identical with Eq.(1.3) of /4/ apart from the notation, provided that we put, in the latter, $t=\sin^2\tau$ and $A=1/n$, and also with Eq.(2.5) of /3/ when $\gamma=0$.

Thus the limiting case in question corresponds to the scheme of a point channel.

Critical mode. Finally we shall consider the critical mode of flow which occurs when $h_0 = H/(2c) - |T|$. Then we have, in accordance with (1.4), $h_2 = -|T|$. Consequently the right end of the cut will rest against the boundary of the tail bay, and a portion of the brine will emerge to its surface. In the w-plane this will correspond to the disappearance of a cut and degeneration of the circular pentagon to a triangle (when $v_0 \leqslant c$), or a quadrangle (when $v_0 > c$), bounded by two ellipsoidal arcs and rectilinear segments orthogonal to them.

In both cases the solutions are obtained from (2.3) by passing to the limit as $k' \to 0$ and from (2.8) at $\alpha = \Lambda/2$ respectively, and take the following form:

$$\frac{dz}{d\tau} = \frac{2Hi}{v_0} \frac{\sin 2(\alpha+\beta)\left(\tau + \frac{\pi}{2}\right)}{\sin 2(\alpha+\beta)\left(\tau - \frac{\pi}{2}\right)}, \quad v_0 \leqslant c \tag{2.10}$$

$$\frac{dz}{d\tau} = \frac{2Hi}{c}\left(1 - i\frac{\sqrt{v_0^2 - c^2}}{v_0} \operatorname{ctg} \pi\tau\right), \quad v_0 > c$$

3. ***Computational scheme and analysis of the results.*** The basic representations (2.3), (2.7) and (2.8) contain an unknown constant, i.e. the modulus k, which can be found with help of the width l of the flood bed. Having found the modulus k, we determine the coordinates of the points of the boundary line AD and the contour of the flood bed with help of the formulas (2.4) and (2.5), as well as the depth of the flood bed d, the difference T between the levels of the head and tail bay, and the filtration rate, from formula (2.2).

Table 1

$c \cdot 10^3$	$T \cdot 10^3$	$d \cdot 10^3$	$Q \cdot 10^4$	$v_0 \cdot 10^4$	$T \cdot 10^3$	$d \cdot 10^3$	$Q \cdot 10^4$
125	227.5	422	556	735	0.77	649	1841
225	89.4	500	643	795	119	483	634
275	71.9	510	647	865	312	297	386

Table 2

l	$T \cdot 10^3$	$d \cdot 10^3$	$Q \cdot 10^4$	l	$T \cdot 10^3$	$d \cdot 10^3$	$Q \cdot 10^4$	l	$T \cdot 10^3$	$d \cdot 10^3$	$Q \cdot 10^4$
1.3	142	459	591	1.32	171	435	544	1.35	218	397	482

Fig.1 shows the smooth contour of the flood bed and the boundary line computed for $H = 0.15$; $h_0 = 1.0$; $c = 0.175$; $v_0 = 0.0805$ and $l = 1.3$ ($l_1 = 1$). Tables 1 and 2 gives the results of the computation showing the effects of the quantities c, v_0 and l on the filtration characteristics T, d and Q in the case when $h_2 > 0$, i.e. $h_0 > H/(2c)$. The parameters c, v_0 and l are varied one at a time, with the remaining parameters are fixed at the values used in Fig.1. The analysis of the dependence of the characteristics sought on the above parameters reduces to the following.

1°. When c increases, i.e. when the dam strength is increased from the side of salt water by a factor of 2.2, d and Q increase by a factor of 1.2. The same changes result when the parameters v_0 and l are increased by 3.6-3.8%, and this indicates how great is the influence of the filtration rate and the width of the flood bed. Moreover, we note that the above changes in the values of v_0 and l lead to somewhat proportional changes in the values of d, Q and also T (when $v_0 \geqslant 0.0805$). This feature was first mentioned in /1/ in connection with the parameters v_0 and H.

2°. The quantity T undergoes the greatest changes, which can be very large. Thus, when the velocity v_0 changes from 0.0735 to 0.0865, T increases by more than 400 times.

3°. The qualitative agreement of the results when the parameters v_0 and l were varied, merits attention. When the parameters decrease, the depth of the flood bed increases sharply together with the rate of flow. The third column of Table 2 shows that another important conclusion is confirmed, which has also been mentioned in /1/, namely that the shorter the flood bed, the thicker it should be for the same value of the velocity v_0.

In conclusion we note that the proposed method can be extended to the case of a rectangular flood bed whose corners are rounded along the curves of the constant rate of filtration. In this case the results of /7/ can again be used for the conformal mapping of the circular hexagon with right angles and two cuts onto a rectangle.

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Translated by L.K.

PMM U.S.S.R.,Vol.54,No.2,pp.289-292,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00
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ESTIMATES OF THE PARAMETERS OF INCREASING PERTURBATIONS IN SHEAR FLOWS OF AN INHOMOGENEOUS MAGNETIZED PLASMA*

N.A. GLAZUNOVA and YU.A. STEPANYANTS

The method of integral relations is used to obtain estimates of the phase velocity and perturbation growth increment in the shear flow of a magnetized plasma, analogous to existing estimates /1, 2/ in the hydrodynamics of stratified fluid, and to refine the results obtained in /3/.

1. We shall start with a well-known system of equations of magnetohydrodynamics for an ideal incompressible fluid of variable density in a gravitational force field /4/:

$$\begin{gathered}\partial_t \mathbf{v} + (\mathbf{v}\nabla)\,\mathbf{v} = \frac{1}{\rho+\rho_1}\left\{-\nabla p + g\rho_1 - \frac{1}{4\pi}[\mathbf{B}\times \operatorname{rot}\mathbf{B}]\right\} \\ \partial_t \mathbf{B} = \operatorname{rot}[\mathbf{v}\times\mathbf{B}],\ \operatorname{div}\mathbf{B} = 0 \\ \partial_t\rho_1 + (\mathbf{v}\nabla)(\rho+\rho_1) = 0,\ \operatorname{div}\mathbf{v} = 0\end{gathered} \tag{1.1}$$

Here p and ρ_1 are the pressure and density perturbations, $\rho(z)$ is the unperturbed density distribution along the vertical, and the remaining notation is traditional.

Let the fluid be contained between two horizontal solid boundaries $z = 0$ and $z = H$. The components of the flow velocity vector and magnetic field strength have, in the unperturbed state, the form $\{U(z), 0, 0\}$, $\{B_0(z), 0, 0\}$. We shall assume that the perturbations of these fields are two-dimensional: $\mathbf{v} = \{u, 0, w\}$, $b = \{b_x, 0, b_z\}$. Linearizing the initial system of equations and seeking the solutions in the form of a product obtained by multiplying the corresponding structural functions depending on z by $\exp\{ik(x-ct)\}$, we reduce the system (1.1) to a single equation for the auxiliary function $f(z) = F(z)[U(z)-c]^n/\rho'$ (a prime denotes a derivative with respect to z), where $F(z)$ is a function defining the structure of the density perturbation along the vertical. We multiply the equation obtained in this manner by a complex conjugate function $\bar{f}(z)$, and integrate the result in z from 0 to H. As a result we arrive at the following integral relation (from now on the limits of integration will be ommitted for simplicity):

$$\begin{gathered}\int \rho[(U-c)^{2(1-n)} - V_A^2(U-c)^{-2n}][|f'|^2 + k^2|f|^2]\,dz + \\ \int\{\rho(U-c)^{-2n}[n(1-n)U'^2 - N^2] + n(U-c)^{1-2n}(\rho U')' - \\ n(U-c)^{-2n-1}U'(\rho V_A^2)' - n\rho V_A^2(U-c)^{-2n-1}U'' + \\ n(n+1)\rho V_A^2(V-c)^{-2(n+1)}U'^2\}|f|^2dz = 0\end{gathered} \tag{1.2}$$

*Prikl. Matem. Mekhan., 54,2,346-349,1990